

1. (a) Since the coordinates of the points P, Q and R are (4, 1, -1), (3, 3, 5) and (1, 0, 2c), respectively, the vectors  $\vec{QR}$  and  $\vec{PR}$  are given by

$$\vec{QR} = -2\vec{i} - 3\vec{j} + (2c - 5)\vec{k} \quad (M1)(A1)$$

$$\vec{PR} = -3\vec{i} - \vec{j} + (2c + 1)\vec{k} \quad (M1)(A1)$$

$$\vec{QR} \text{ is perpendicular to } \vec{PR} \text{ if and only if } \vec{QR} \cdot \vec{PR} = 0 \text{ i.e. } 6 + 3 + (2c - 5)(2c + 1) = 0 \quad (M1)$$

$$\Rightarrow 4c^2 - 8c + 4 = 0$$

$$\Rightarrow (c - 1)^2 = 0 \quad (M1)$$

$$\Rightarrow c = 1 \quad (A1)$$

- (b)  $\vec{PR} = -3\vec{i} - \vec{j} + 3\vec{k}$ ,  $\vec{PS} = -3\vec{i} + 3\vec{k}$  (M1)(M1)

$$\vec{PS} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 3 \\ -3 & -1 & 3 \end{vmatrix} \quad (M1)$$

$$= 3\vec{i} + 3\vec{k} \quad (A1)$$

- (c) The parametric equation of a line  $l$  which passes through the point (3, 3, 5) and is parallel to the vector  $\vec{PR}$  is given by

$$\begin{aligned} \vec{r} &= (3\vec{i} + 3\vec{j} + 5\vec{k}) + t(-3\vec{i} - \vec{j} + 3\vec{k}) \\ &= 3(1 - t)\vec{i} + (3 - t)\vec{j} + (5 + 3t)\vec{k} \quad (-\infty < t < \infty) \end{aligned} \quad \begin{array}{l} (M1)(M1) \\ (A1) \end{array}$$

**Note:** If  $-\infty < t < \infty$  is not mentioned, do not penalise.  
Also note that some candidates may give the parametric equation of the line in the form  $x = 3(1 - t)$ ,  $y = 3 - t$ ,  $z = (5 + 3t)$ ,  $-\infty < t < \infty$

continued...

Question 1 continued

- (d) Let  $P_1$  and  $P_2$  be points on the line  $l$  corresponding to  $t = 0$  and  $t = 1$ , respectively.

Hence, for  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ ,

$$x = 3(1-t), y = (3-t) \text{ and } z = 5+3t.$$

Putting  $t = 0$  and  $t = 1$ , we get the coordinates of points  $P_1$  and  $P_2$  as  $(3, 3, 5)$  and  $(0, 2, 8)$ , respectively.

Vectors  $\vec{SP}_1$  and  $\vec{SP}_2$  are given by  $\vec{SP}_1 = 2\vec{i} + 2\vec{j} + 3\vec{k}$  and  $\vec{SP}_2 = -\vec{i} + \vec{j} + 6\vec{k}$ . (M1)

A vector perpendicular to both  $\vec{SP}_1$  and  $\vec{SP}_2$  is  $\vec{SP}_1 \times \vec{SP}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 3 \\ -1 & 1 & 6 \end{vmatrix}$  (M1)

$$= 9\vec{i} - 15\vec{j} + 4\vec{k}$$
 (A1)

Let  $T(x, y, z)$  be any point of the plane  $\pi$ . Since  $S = (1, 1, 2)$ ,

$\vec{ST} = (x-1)\vec{i} + (y-1)\vec{j} + (z-2)\vec{k}$  is a vector in  $\pi$ . Hence  $\vec{ST} \times \vec{n} = 0$

$$\text{i.e. } 9(x-1) - 15(y-1) + 4(z-2) = 0 \Rightarrow 9x - 15y + 4z - 2 = 0$$
 (A1)

OR

$$\vec{QS} = (1-3)\vec{i} + (1-3)\vec{j} + (2-5)\vec{k}$$
 (M1)

$$= -2\vec{i} - 2\vec{j} - 3\vec{k}$$
 (M1)

The equation of the plane containing the line  $l$  and passing through the point  $S$  is determined by  $l$  and the vector  $\vec{SQ}$ . Hence, the equation is:

$$\vec{r} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix}$$
 (M1)(A1)

(e) Shortest distance  $\frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{\begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -15 \\ 4 \end{pmatrix}}{\sqrt{322}}$  (M1)(M1)

$$= \frac{15}{\sqrt{322}}$$
 (M1)(A1)

OR

The distance of  $P$  from  $\pi$  is:

$$\frac{|9(4) - 15(1) + 4(-1) - 2|}{\sqrt{9^2 + 15^2 + 4^2}}$$
 (M1)(A1)

$$= \frac{15}{\sqrt{322}}$$
 (M1)(A1)

**Note:** Accept 0.836 (3 s.f.)

2. (i) Let the arithmetic sequence be written as  $a, a+d, a+2d, \dots$

$$\text{Then } \frac{a+4d}{a+11d} = \frac{6}{13} \quad (M1)$$

$$\text{So } 13a+52d=6a+66d \Rightarrow 7a=14d \quad (M1)$$

$$\Rightarrow a=2d. \quad (A1)$$

Since each term is positive, both  $a$  and  $d$  are positive. We are given  $a(a+2d)=32$ , setting  $a=2d$ , we get  $2d(2d+2d)=8d^2=32$ . (M1)

$$\Rightarrow d=\pm 2. \quad (A1)$$

Hence,  $d=2$  and  $a=4$  and sum to 100 terms of this sequence is

$$\frac{100}{2}\{(2)(4)+(100-1)2\}. \quad (M1)$$

$$=10300 \quad (A1)$$

- (ii) (a) Since  $\omega$  is a complex number which satisfies  $\omega^3-1=0$ ,  $\omega \neq 1$ . Hence,

$$1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}=0. \quad (M1)(A1)$$

$$(b) (\omega x + \omega^2 y)(\omega^2 x + \omega y) = \omega^3 x^2 + \omega^4 yx + \omega^2 xy + \omega^3 y^2. \quad (M1)$$

Using  $\omega^3=1$  and  $\omega^4=\omega$ , we get, (M1)

$$(\omega x + \omega^2 y)(\omega^2 x + \omega y) = (x^2 + y^2) + (\omega^2 + \omega)xy \quad (M1)$$

$$= x^2 + y^2 - xy, \text{ (Since } 1+\omega+\omega^2=0) \quad (A1)$$

- (iii) Let  $S(n)$  be the statement:  $2^{2n}-3n-1$  is divisible by 9.

Since  $2^2-3-1=0$ ,  $S(1)$  is true. (C1)

Assume as the induction hypothesis  $S(k)$  i.e.  $2^{2k}-3k-1$  is divisible by 9. (C1)

We shall show that  $S(k+1)$  is true.

$$S(k+1) = 2^{2(k+1)} - 3(k+1) - 1 \quad (M1)$$

$$= 4(2^{2k}) - 3k - 4 \quad (M1)$$

$$= 4(2^{2k} - 3k - 1) + 9k \quad (A1)$$

By the induction hypothesis  $2^{2k}-3k-1$  is divisible by 9. Since  $9k$  is also divisible by 9,  $S(k+1)$  is true.

Thus, by mathematical induction  $S(n)$  is true,  $n=1, 2, \dots$  (M1)

(R1)

3. (i) Let  $D$  be the event that the patient has the disease and  $S$  be the event that the new blood test shows that the patient has the disease. Let  $D'$  be the complement of  $D$ , i.e. the patient does not have the disease.

Now the given probabilities can be written as

$$p(S|D) = 0.99, p(D) = 0.0001, p(S|D') = 0.05.$$

(A1)(A1)(A1)

Since the blood test shows that the patient has the disease, we are required to find  $p(D|S)$ .

By Bayes' theorem,

$$p(D|S) = \frac{p(S|D)p(D)}{p(S|D)p(D) + p(S|D')p(D')} \quad (M1)$$

$$= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.05)(1 - 0.0001)} \quad (M1)$$

$$= 0.001976... = 0.00198 \text{ (3 s.f.)} \quad (A1)$$

**Note:** Some candidates may use a tree diagram. Please check the steps and award marks accordingly.

- (ii) Let  $n$  = number of chips = 1000,

$p$  = the probability that a randomly chosen chip is defective = 0.02.

Hence, the mean  $np = (1000)(0.02) = 20$  and the

variance  $= np(1-p) = (1000)(0.02)(0.98) = 19.6$ .

(A1)(A1)

Suppose  $X$  is the normal random variable that approximates the binomial distribution.

The  $X \approx N(20, 19.6)$ .

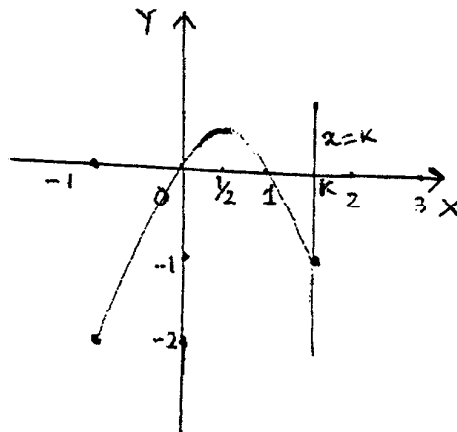
(M1)(M1)

$$\begin{aligned} \text{Thus } p(19.5 \leq X \leq 30.5) &= p\left(\frac{19.5 - 20}{\sqrt{19.6}} \leq Z \leq \frac{30.5 - 20}{\sqrt{19.6}}\right) \\ &= p(-0.11 \leq Z \leq 2.37) \\ &= 0.5349 \end{aligned} \quad (M1)(M1)$$

(A1)

**Notes:** Some may write  $p(-0.113 \leq Z \leq 2.372) = 0.5362$ . Award full marks.  
Accept answers correct to 3 s.f.

4. (i) (a)



Notes: Award (A1) for the correct shape  
(A1) for both end points  
(A1) for the line  $x = k$

(b) Required area =  $\int_0^1 (x - x^2) dx - \int_1^k (x - x^2) dx$

(M1)

$$= \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 - \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_1^k$$

(A1)

$$= \frac{1}{6} - \frac{k^2}{2} + \frac{k^3}{3} + \frac{1}{6} = \frac{1}{3} + \frac{k^3}{3} - \frac{k^2}{2}$$

(M1)

$$= \frac{1}{6} (2 + 2k^3 - 3k^2)$$

(A1)

continued...

Question 4 continued

(ii) (a)  $g(t) = \frac{\ln t}{\sqrt{t}}$ . So  $g'(t) = \frac{2 - \ln t}{2t^{3/2}}$ . (M1)(A1)

Hence,  $g'(t) \geq 0$  when  $2 \geq \ln t$  or  $\ln t \leq 2$  or  $t \leq e^2$ . (M1)

Since the domain of  $g(t)$  is  $\{t: t > 0\}$ ,  $g'(t) \geq 0$  when  $0 < t \leq e^2$ . (A1)

(b) Since  $g'(t) = \frac{2 - \ln t}{2t^{3/2}}$ ,  $g''(t) = \frac{-2\sqrt{t} - 3\sqrt{t}(2 - \ln t)}{4t^3}$  (M2)

$$= -\frac{\sqrt{t}[8 - 3\ln t]}{4t^3} \quad (A1)$$

Hence  $g''(t) > 0$  when  $8 - 3\ln t < 0$  i.e.  $t > e^{8/3}$ . (M1)

Similarly,  $g''(t) < 0$  when  $0 < t < e^{8/3}$ . (M1)

Hence there is a point of inflexion when  $t = e^{8/3}$ . (A1)

(c)  $g''(t) = 0$  when  $t = 0$  or  $8 = 3\ln t$ .  
Since, the domain of  $g$  is  $\{t: t > 0\}$ ,  $g''(t) = 0$  when  $t = e^{8/3}$ . (M1)

Since  $g''(t) > 0$  when  $t > e^{8/3}$  and  $g''(t) < 0$  when  $t < e^{8/3}$ , (M1)

$\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$  is the point of inflexion. The required value of  $t$  is  $e^{8/3}$ . (A1)

**Note:** Award (A1) for evaluating  $t$  as  $e^{8/3}$ .

(d)  $g'(t) = 0$  when  $\ln t = 2$  or  $t = e^2$ . (M1)

Also  $g''(e^2) = -\frac{\sqrt{e^2}[8 - \ln e^2]}{4(e^2)^3} = -\frac{6e}{4e^6} < 0$ . (M1)

Hence  $t^* = e^2$ . (A1)

(e) At  $(t^*, g(t^*))$  the tangent is horizontal. (M1)

So the normal at the point  $(t^*, g(t^*))$  is the line  $t = t^*$ . (M1)

Thus, it meets the  $t$  axis at the point  $t = t^* = e^2$  and hence the point is  $(e^2, 0)$ . (A1)

5. (i) (a)  $S$  is the group of permutations of  $\{1, 2, 3\}$  under the composition of permutation. Since  $3! = 6$ , order of  $S = 6$ . (M1)  
(R1)

(b) Members of  $S$  are  $p_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ ,  $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ ,  $p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ , (AG)

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}. \quad (A2)$$

**Note:** Award (A2) for 3 correct permutations;  
(A1) for 2 correct permutations;  
(A0) for 1 correct permutation.

$$\text{Since } p_3 \circ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\text{and } p_4 \circ p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad (M1)$$

$$p_3 \circ p_4 \neq p_4 \circ p_3 \quad (R1)$$

**Note:** There are other possibilities to show that the group is not Abelian.

(c)  $p_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = p_2$

$$p_1^3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = p_0. \quad (M1)$$

(Note that  $p_0$  is the identity of the group  $S$ .)

Hence  $\{p_0, p_1, p_2\}$  form a cyclic group of order 3 under the binary operation of composition of permutations. (R1)

**Note:** Some candidates may write  $\{p_0, p_1, p_2\}$  is a subgroup of order 3, (award (A1)), and write the following table, (award (R1)):

$\circ$	$p_0$	$p_1$	$p_2$
$p_0$	$p_0$	$p_1$	$p_2$
$p_1$	$p_1$	$p_2$	$p_0$
$p_2$	$p_2$	$p_0$	$p_1$

continued...

Question 5 continued

(ii) (a) Let  $A = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$ .

Let  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$  and  $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$  be two elements of  $A$ . Then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix} \quad (M1)$$

$$= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \text{ if we write } \alpha = ac - bd, \beta = ad + bc. \quad (R1)$$

Since  $\alpha, \beta$  are real numbers  $A$  is closed under matrix multiplication.

Associativity follows from the fact that matrix multiplication is an associative binary operation on the collection of all  $2 \times 2$  matrices and the fact that  $A$  is closed with respect to the binary operation of multiplication of matrices. (R1)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in A \text{ and is the identity since } \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (M1)(M1)$$

for any element  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$  of  $A$ . (R1)

Since  $a \neq 0$ , the determinant  $\begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2 + b^2 \neq 0$ . (M1)

Hence the matrix  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$  has an inverse, which is  $\frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$  (R1)(A1)

since  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \left( \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and

$$\frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (R1)$$

**Note:** Some candidates may use  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  to find the inverse of  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ . Award marks as appropriate.

continued...



Question 5 (ii) continued

$$(b) \quad M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$M$  contains the identity  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

(A1)

The following results show that the set  $M$  is closed with respect to matrix multiplication as the binary operation.

(R1)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(M2)

Since the product of each element with itself is the identity, every element of  $M$  is its own inverse. Hence  $M$  is a subgroup of  $A$ .

(R1)

- (c) The order of the group  $M$  is 4. If it had a subgroup of order 3, then by the corollary of the Lagrange's theorem, 3 must be a divisor of 4. Since it is not true,  $M$  can not have a subgroup of order 3.

(M1)

(R1)

- (iii) (a) Let  $(G, \circ)$  and  $(H, \bullet)$  be two groups. They are said to be isomorphic if there exists a one-to-one transformation  $f: G \rightarrow H$  which is surjective (onto) with the property that for all  $x, y \in G$ ,  $f(x \circ y) = f(x) \bullet f(y)$ .

(C1)

(C1)

**Note:** Some candidates may say that the groups  $(G, \circ)$  and  $(H, \bullet)$  are isomorphic if they have the same Cayley table (or group table). In that case award (C1).

- (b) Since  $f: G \rightarrow H$ ,  $f(x) \in H$  for some  $x \in G$ .  
 Since  $e'$  is the identity element in  $H$ ,  
 $e' \bullet f(x) = f(x) = f(x \circ e) = f(e) \bullet f(x)$ .  
 By the right cancellation law,  $e' = f(e)$ .

(M1)

(M1)(A1)

(R1)

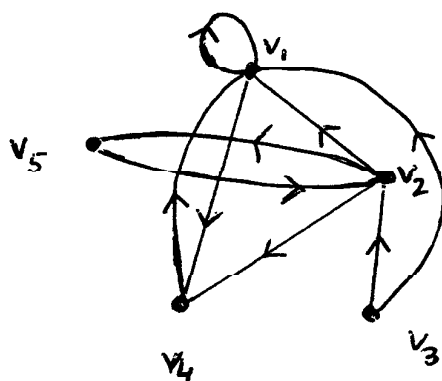
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Question 5 (iii) continued

- (c) Suppose  $G = \langle a \rangle$ , the cyclic group generated by  $a$ , i.e.  $n$  is the smallest positive integer such that  $a^n = e$ , the identity in  $G$ . (C1)  
 Let  $f: G \rightarrow H$  be an isomorphism. Let  $f(a) = b \in H$ .  
 $f(a^2) = f(a \circ a) = f(a) \bullet f(a) = (f(a))^2$ . In general  $f(a^n) = (f(a))^n$ . (M1)(A1)  
 By (iii) (b)  $(f(a))^n = e'$ , the identity in  $H$ . Hence  $b^n = e'$  and consequently  $H$  is a cyclic group of order  $n$  with generator  $b$ . (R1)
- (d)  $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$ . Let  $\oplus_4$  be the binary operation. (M1)  
 $[1] \oplus_4 [1] = [2], [1] \oplus_4 [2] = [3], [1] \oplus_4 [3] = [0]$  (M1)(A1)  
 Hence  $[1]$  is a generator of the group  $(\mathbb{Z}_4, \oplus_4)$  and  $\mathbb{Z}_4$  is cyclic.  
 Since each element of  $M$  is of order 2,  $M$  is not cyclic. (M1)  
 Hence  $\mathbb{Z}_4$  is not isomorphic to  $M$ . (R1)

6. (i) (a) If  $G$  is a directed graph with  $n$  vertices, then an adjacency matrix of  $G$  is an  $n \times n$  matrix whose  $i, j$  entry is a number different from zero if there is a directed edge from the vertex  $v_i$  to the vertex  $v_j$ , and zero if there is no directed edge connecting  $v_i$  and  $v_j$ . (C2)
- (b) The sum of the entries in row  $i$  of the adjacency matrix of the directed graph  $G$  is equal to the outdegree of the vertex  $v_i$  of  $G$ , i.e. the number of directed edges from vertex  $v_i$ . (C2)
- (c) The sum of the entries in column  $j$  equals the indegree of the vertex  $j$  i.e. the number of directed edges to vertex  $v_j$ . (C2)

(ii) (a)



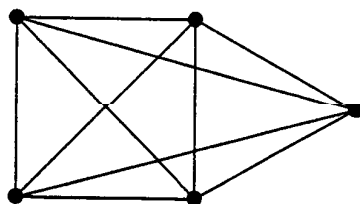
Graph H

(A3)

**Note:** Award (A3) for 6 correct, (A2) for 5 correct; (A1) for 4 correct, (A0) for 3 or fewer correct

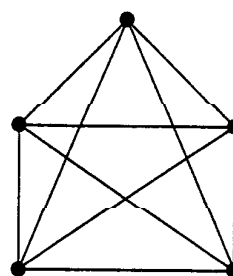
- (b) Since the element in the second row and the fourth column of  $A^2$  is 1, there is only one path of length 2 from  $v_2$  to  $v_4$ . (M1)(R1)
- (iii) (a) (i) A complete graph  $K_5$  is a graph on 5 vertices where every vertex is connected to every other vertex. (A2)

Example:



$K_5$

or



$K_5$

(A1)

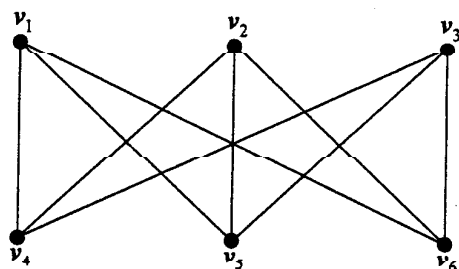
continued...

Question 6 (iii) continued

- (ii) In  $\kappa_{3,3}$  vertices are divided into two disjoint sets  $A$  and  $B$ , each having three vertices such that no two vertices in  $A$  are adjacent and no two vertices in  $B$  are adjacent, but every vertex of  $A$  is adjacent to every vertex of  $B$ .

(A1)

Example:

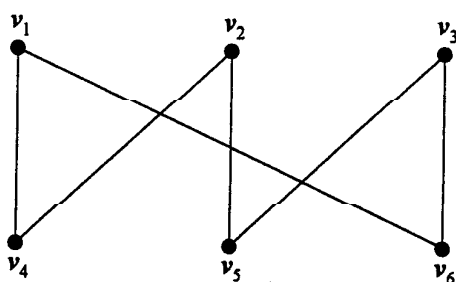


$\kappa_{3,3}$

(A1)

- (b) To find a Hamiltonian circuit in  $\kappa_{3,3}$ , we show that there is a circuit which goes through each vertex exactly once. Such a circuit is shown below:

(M1)



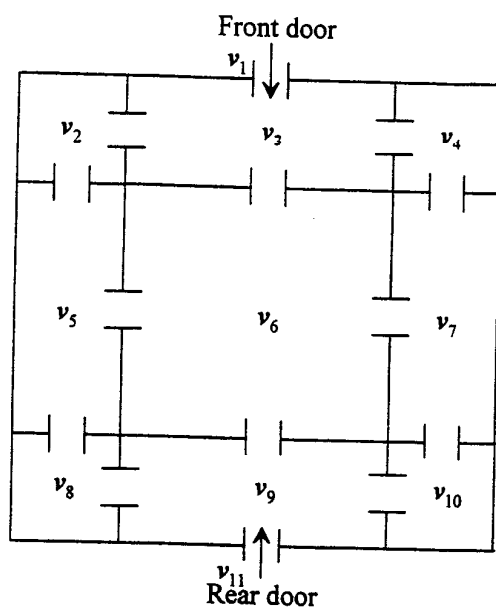
(M1)(A1)

**Note:** Please check the circuit drawn by the candidate and award (M2)(A1) for a correct answer.

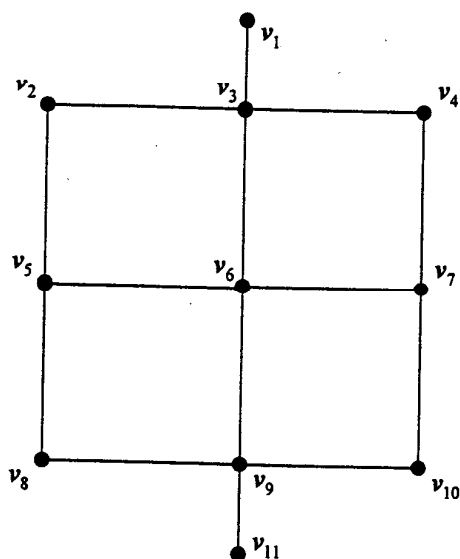
Also note that some candidates may not show the Hamiltonian circuit but mention that each vertex of  $\kappa_{3,3}$  has degree  $3 = \frac{6}{2}$ . Hence there is a Hamiltonian circuit. In this case award (M2)(A1).

continued...

(iv)



(M1)



**(MI)(AI)**

**(R1)**

**(R1)**

*continued...*

Question 6 continued

- (v) (a) The degree sequence yields the number of edges at each vertex and an isomorphism preserves adjacency of vertices. (M1)  
Hence, if two groups are isomorphic and the degree sequences are not the same then the isomorphism can not preserve the adjacency of vertices contradicting our observation. (M1)  
Thus an isomorphism must preserve degree sequences. (R1)
- (b) The degree sequences of the graphs  $G$  and  $H$  are 2, 3, 3, 3, 3, 4 and 2, 3, 3, 3, 5, 2 respectively. (M1)(M1)  
Since they are not the same, the two graphs are not isomorphic by (a). (R1)

- (vi) We start with point  $A$  and write  $S$  as the set of vertices and  $T$  as the set of edges. The weights on each edge will be used in applying Prim's algorithm. Initially,  $S = \{A\}$ ,  $T = \emptyset$ . In each subsequent stage, we shall update  $S$  and  $T$ .

<u>Step 1:</u>	Add edge $h$ :	So $S = \{A, D\}$ ,	$T = \{h\}$	
<u>Step 2:</u>	Add edge $e$ :	So $S = \{A, D, E\}$	$T = \{h, e\}$	
<u>Step 3:</u>	Add edge $d$ :	Then $S = \{A, D, E, F\}$	$T = \{h, e, d\}$	
<u>Step 4:</u>	Add edge $a$ :	Then $S = \{A, D, E, F, B\}$	$T = \{h, e, d, a\}$	
<u>Step 5:</u>	Add edge $i$ :	Then $S = \{A, D, E, F, B, G\}$	$T = \{h, e, d, a, i\}$	
<u>Step 6:</u>	Add edge $g$ :	Then $S = \{A, D, E, F, B, G, C\}$	$T = \{h, e, d, a, i, g\}$	(M3)(A3)

**Note:** Award (M3)(A3) for all 6 correct, (M3)(A2) for 5 correct;  
(M2)(A2) for 4 correct, (M2)(A1) for 3 correct;  
(M1)(A1) for 2 correct, (M1)(A0) for 1 correct.

Now  $S$  has all the vertices and the minimal spanning tree is obtained by going through the vertices  $A \rightarrow D \rightarrow E \rightarrow F \rightarrow B \rightarrow G \rightarrow C$ .

The weight of the edges in  $T$  is  $5 + 3 + 5 + 7 + 5 + 6 = 31$

(M1)  
(R1)

7. (i) (a) Let  $X$  denote the number of flaws in one metre of the wire. Then  $E(X) = 2.3$  flaws and  $P(X = 2) = e^{-2.3} \frac{(2.3)^2}{2!}$   
 $= 0.265$ . (M1)(M1)

(A1)

- (b) Let  $Y$  denote the number of flaws in two metres of wire. Then  $Y$  has a Poisson distribution with mean  $E(Y) = 2 \times 2.3 = 4.6$  flaws for 2 metres.

(M1)

Hence,  $P(Y \geq 1) = 1 - P(Y = 0) = 1 - e^{-4.6}$   
 $= 0.9899$

(M1)

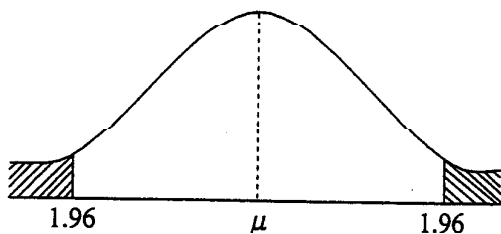
(A1)

**Note:** Accept 0.990 (3 s.f.), or  $1 - e^{-4.6}$ .

- (ii) Let  $\bar{x}$  be the sample mean,  $\mu$  be the population mean and  $\sigma$  be the population standard deviation.

Hence,  $\sigma_{\bar{x}}$ , the standard error is given by  $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{4}{\sqrt{n}}$ , where  $n$  is the sample size which is required. (M1)(A1)

The 95 % confidence interval of the population mean  $\mu$  is given by  $(\bar{x} - \mu) \pm 1.96\sigma_{\bar{x}}$ . (M1)(A1)



Hence, we need to find  $n$  so that  $(\sigma_{\bar{x}})(1.96) \leq 0.25$  (M1)(R1)

$$\Rightarrow \left( \frac{4}{\sqrt{n}} \right) (1.96) \leq 0.25$$

$$\Rightarrow \left( \frac{1}{\sqrt{n}} \right) \leq \frac{0.25}{4(1.96)} \Rightarrow \sqrt{n} \geq \frac{4(1.96)}{0.25} = 31.36$$
 (M1)(M1)

Hence,  $n \geq 983.44$  (A1)

The sample size required is 984. (R1)

**Note:** Award (M2)(R2) if candidates write: "We need to find  $n$  such that  $\left( \frac{4}{\sqrt{n}} \right) (1.96) \leq 0.25$ ."

continued...

Question 7 continued

- (iii) Let  $\mu_1$  and  $\mu_2$  be the weekly mean wage of science and humanities students respectively. The hypotheses to be tested are

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 - \mu_2 \neq 0$$

(A1)

(A1)

It is a two-tailed test. The test statistic is  $\bar{x}_1 - \bar{x}_2$ . The two random variables are independent i.e. probability of selection of a science student is not affected by the selection of a humanity student.

$$\mu_{\bar{x}_1 - \bar{x}_2} = 0, \sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where  $\sigma_1, \sigma_2$  are the population standard deviations of science and humanity students respectively. Since the samples are large one can approximate  $\sigma_1, \sigma_2$  by  $s_1, s_2$ , respectively. The resulting estimated standard error is given by

$$s_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

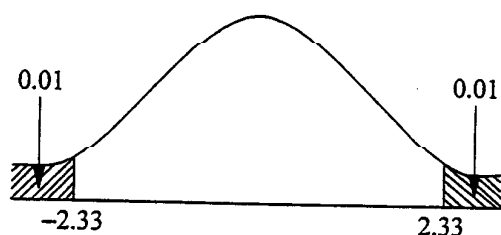
(M2)

$$= \sqrt{\frac{4^2}{100} + \frac{(4.5)^2}{200}}$$

(A1)

$$= 0.51...$$

(A1)



From the normal distribution table we obtain the decision rule that if the computed value of  $z > 2.33$  or  $z < -2.33$  then we reject  $H_0$ .

(M1)(A1)

$$\text{But } z = \frac{\bar{x}_1 - \bar{x}_2}{s_{\bar{x}_1 - \bar{x}_2}}$$

(M1)

$$= \frac{\$120.5 - 115}{0.51}$$

(M1)

$$= 10.784 > 2.33$$

(A1)

So we reject  $H_0$ .

(R1)

Thus, we conclude the wages of science and humanity students are not the same.

**Note:** An alternative approach by calculating critical values for a rejection region is possible. Please check the working and award marks accordingly.

continued...



Question 7 continued

- (iv)  $H_0$ : the cost of the automobile is independent of the number of complaints.  
 $H_1$ : the cost of an automobile is not independent of the number of complaints.

(A1)

The observed frequencies are first totalled, and then the expected frequencies under  $H_0$  are calculated from the formula:

$$\text{expected frequency} = \frac{(\text{row total})(\text{column total})}{\text{grand total}}$$

Cost	Number of complaints			Total
	$\leq 5$	6-10	$\geq 11$	
$\geq \$ 30001$	<b>60</b>	<b>120</b>	<b>20</b>	200
$\$ 15001 - \$ 30000$	<b>138</b>	<b>276</b>	<b>46</b>	460
$\leq \$ 15000$	<b>102</b>	<b>204</b>	<b>34</b>	340
Total	300	600	100	1000

(A4)

Table of expected frequencies

**Note:** Award (A4) for 9 correct bold entries  
 (A3) for 7 or 8 correct bold entries  
 (A2) for 5 or 6 correct bold entries  
 (A1) for 4 correct bold entries  
 (A0) for 3 or less bold entries

$$\chi^2 = \sum \left\{ \frac{(f_o - f_e)^2}{f_e} \right\}$$

(M1)

We shall calculate  $\chi^2$  from a table where we shall list  $f_o, f_e, (f_o - f_e)^2, \frac{(f_o - f_e)^2}{f_e}$ , where  $f_o$  is the observed number of complaints and  $f_e$  is the expected number of complaints.

continued...

Question 7 (iv) continued

	$f_o$	$f_e$	$f_o - f_e$	$(f_o - f_e)^2$	$\frac{(f_o - f_e)^2}{f_e}$
	100	60	40	1600	26.67
	90	120	-30	900	7.5
	10	20	-10	100	5.0
	150	138	12	144	1.04
	260	276	-16	256	0.93
	50	46	4	16	0.35
	50	102	52	2704	26.51
	250	204	46	2116	10.37
	40	34	6	36	1.06
Total	1000	1000			$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 79.43$

(43)

Calculation of  $\chi^2$

$$v = \text{number of degrees of freedom} = (3 - 1)(3 - 1) = 4$$

(M1)

At the 5 % level of significance,  $\chi^2_{0.05}$  with four degrees of freedom,  $\chi^2_{0.05} = 9.49$

(A1)

Decision rule: If  $\chi^2 > 9.49$ , reject  $H_0$ . If  $\chi^2 \leq 9.49$ , then accept  $H_0$ .

Our computed  $\chi^2 = 79.43 > 9.49$  and hence according to our decision rule, we reject  $H_0$ .

Conclusion: The cost of the automobile is not independent of the number of complaints.

(R1)

**Note:** Award (M1)(R1) for stating:  $\chi^2_{0.05} = 9.49 < 79.43$ , hence reject  $H_0$ .

8. (i) (a) Let  $f(x) = x^3 - 3x - 5$ . Hence  $f(1) = -7 < 0$  and  $f(3) = 13 > 0$ . (M1)  
Hence, by the intermediate value theorem there is one zero of  $f(x) = x^3 - 3x - 5$  in the interval  $1 \leq x \leq 3$ . (A1)

(b)  $f(x) = x^3 - 3x - 5$   
 $f'(x) = 3x^2 - 3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n - 5}{3x_n^2 - 3}$$

(M1)(A1)

$n$	$x_n$	$x_{n+1}$	$ x_{n+1} - x_n $
0	2	2.333333	0.333333
1	2.333333	2.280556	0.052777
2	2.280556	2.279020	0.001536
3	2.279020	2.279019	$1 \times 10^{-6}$
4	2.279019	2.279019	$1 \times 10^{-6}$

(A3)

**Note:** Award (A3) for 5 correct bold entries  
(A2) for 4 correct bold entries  
(A1) for 2 or 3 correct bold entries  
(A0) for 0 or 1 correct bold entries

Since we need accuracy  $10^{-5}$ , we need  $|x_n - x_{n+1}| < 5 \times 10^{-6}$ .  
So  $x = 2.27902$  is the solution of  $f(x) \approx 0$ .

(R1)

- (ii) (a) Formula for the trapezium rule with  $n = 6$  gives,

$$\int_0^1 g(x) dx = \frac{1}{12} [g(x_0) + 2g(x_1) + 2g(x_2) + 2g(x_3) + 2g(x_4) + 2g(x_5) + g(x_6)]$$

(M2)

$$= 0.745$$

(A1)

Formula for Simpson's rule gives,

$$\int_0^1 g(x) dx = \frac{1}{18} [g(x_0) + 4g(x_1) + 2g(x_2) + 4g(x_3) + 2g(x_4) + 4g(x_5) + g(x_6)]$$

(M2)

$$= 0.747$$

(A1)

- (b)  $|\text{Error by the Simpson's rule}| \leq \max_{0 \leq x \leq 1} |g^{(4)}(x)| \left( \frac{1}{180} \right) \left( \frac{1}{6} \right)^4$  (M1)(A1)

continued...

Question 8 (ii) continued

- (c) Let  $S_n(g)$  be the approximation to  $\int_0^1 g(x)dx$  by Simpson's rule over  $n$  intervals of equal length.

$$\text{So } \left| \int_0^1 g(x)dx - S_n(g) \right| \leq \frac{6}{(180)n^4} \quad (M1)$$

$$= \frac{1}{30n^4}, \quad (A1)$$

Since  $|g^{(4)}(x)| \leq 6$  for  $0 \leq x \leq 1$ .

To be correct to 5 d.p., we need to find  $n$  so that (M1)

$$\frac{1}{30n^4} \leq 5 \times 10^{-6} \quad (M1)$$

$$\text{i.e. } 30n^4 \geq \frac{10^6}{5} \text{ or } n^4 \geq \frac{10^6}{150} \text{ or } n \geq \sqrt[4]{\frac{10^6}{150}} = 9.036 \quad (M1)$$

Hence, we need 10 intervals.

- (iii) (a) If  $g(x)$  is continuous for  $a \leq x \leq b$  and differentiable for  $a < x < b$ , then there exists a  $\xi$  in  $a < \xi < b$  so that (A1)  
 $g(b) - g(a) = (b - a)g'(\xi), a < \xi < b.$  (A1)

- (b) By the mean value theorem  $h(x) - h(0) = h'(c)(x - 0)$ , for  $0 < c < 7$ . (M1)

Therefore  $|h(x) - h(0)| \leq 10x$ , for  $0 \leq x \leq 7$ , since  $|h'(x)| \leq 10$ . (A1)

Thus  $-70 + h(0) \leq h(x) \leq h(0) + 70$  (M1)

$$\Rightarrow -70 - 4 \leq h(x) \leq -4 + 70$$

$$\Rightarrow -74 \leq h(x) \leq 66$$

Hence  $h(x) \geq -74$ , for  $0 \leq x \leq 7$ . (A1)

- (iv) (a) Let  $u_k = \frac{k+1}{3^k}$ .

$$\text{Then } \lim_{k \rightarrow \infty} \frac{u_{k+1}}{u_k} = \lim_{k \rightarrow \infty} \frac{(k+2)}{3^{k+1}} \times \frac{3^k}{(k+1)} = \frac{1}{3} < 1. \quad (M2)$$

and hence the series converges by the ratio test. (R1)

continued...

Question 8 (iv) continued

(b) Let  $f(x) = \frac{1}{x(\ln x)^3}$

Since  $x(\ln x)^3$  is an increasing function of  $x$  for  $x > 1$ ,  $f(x)$  is a decreasing function of  $x$  for  $x > 1$ .

(M1)

Further,  $f(k) = \frac{1}{k(\ln k)^3}$

Hence, by the integral test,  $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$  converges or diverges according to the convergence or divergence of  $\int_2^{\infty} \frac{1 dx}{x(\ln x)^3} = \lim_{R \rightarrow \infty} \int_2^R \frac{dx}{x(\ln x)^3}$

(M1)

$$= \lim_{R \rightarrow \infty} \left[ \frac{(\ln x)^{-2}}{-2} \right]_2^R = \lim_{R \rightarrow \infty} \left\{ -\frac{1}{2(\ln R)^2} + \frac{1}{2(\ln 2)^2} \right\} = \frac{1}{2(\ln 2)^2}$$

(A1)

So the series converges, by the integral test.

(R1)

(c)  $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 + 1}$  is an alternating series.

When  $f(x) = \frac{x}{x^2 + 1}$ ,  $f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} \leq 0$ ,

(M1)(A1)

when  $x \geq 1$ , the sequence  $\left\{ \frac{k}{k^2 + 1} \right\}$  is a decreasing sequence.

Also  $\lim_{k \rightarrow \infty} \frac{k}{k^2 + 1} = \lim_{k \rightarrow \infty} \frac{1}{k + \frac{1}{k}} = 0$ .

(M1)(A1)

Hence the series converges by the alternating series test.

(R1)